Moment-of-Fluid on Cartesian grids

Interface representation & reconstruction

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Introduction: representing multiphase flow

How to represent multiphase flow on meshes?

Physics
- Multiphase flow
- Immiscible fluids
- Mass conservation
- ...

Numerical representation
- Volume-of-fluid
- Level-set
- Front tracking
- Moment-of-fluid
- ...

Source: BlueWater by trancedman, DeviantArt
1. Introduction

2. Interface reconstruction

3. Advection

4. Numerical results

5. Perspectives
Example: Volume-of-Fluid

Representation ≠ Reconstruction

'True' interface

VOF representation

PLIC reconstruction

VOF: Volume-of-Fluid
PLIC: Piecewise Linear Interface Construction
VOF-PLIC formulation

Original data
- $\Omega$ polygonal cell of vertices $\{p_1, \ldots, p_n\}$
- $\omega_1^*$ portion of fluid 1 in the cell $\Omega$

Representation
- $|\omega_1^*|$ volume of fluid 1

Reconstruction
- $\omega_1^\ell(\phi)$ polygonal approximation of $\omega_1^*$
- $|\omega_1^\ell(\phi)| = |\omega_1^*|$ (volume conservation)
- $\Gamma^\ell(\phi) = \partial \Omega \setminus \partial \omega_1^\ell(\phi)$ interface
- $n(\phi)$ interface normal
- $\phi$ angle with the horizontal axis

Parametrization $\Gamma^\ell = \{x \in \Omega/ x \cdot n = d\}$
- $n$ interface normal
- $d$ distance to the origin
**VOF-PLIC formulation**

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- \( d \) distance to the origin
How to find $d$ and $n$?

What do we want?

→ Find a linear interface as close as possible of the original interface

- Problem: 1 constraint, 2 unknown
- Hard part: finding $n$

Classic methods that use information of surrounding cells:

- Gradient of volume fraction
- Least-square
- LVIRA, ELVIRA
- ...

- Easy part: if we know $n$, the distance $d$ can be deduced from the volume constraint

→ Flood algorithm
Flood algorithm

Initial condition
- Flood direction $n$
- Volume of fluid $|\omega^*| = |\Omega|/2$
- $p_4$ first point
- $\xi_4$ first distance

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Sequential algorithm (find $\xi^*$)
1. $\xi^* \notin [\xi_4, \xi_3]$
2. $\xi^* \notin [\xi_3, \xi_5]$
3. $\xi^* \notin [\xi_5, \xi_2]$
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Result
- $\xi^*$ given by quadratic interpolation

Limitations of VOF methods

- Original interface is piecewise linear in each cell
- Reconstruction should be exact!

Problem

The volume fraction is insufficient to make a cell-wise reconstruction

Idea

Add information to have a local (cell-wise) reconstruction → Moment of Fluid
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Add information to have a *local* (cell-wise) reconstruction $\rightarrow$ Moment of Fluid
1 Introduction

2 Interface reconstruction

3 Advection

4 Numerical results

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Moment-of-fluid: Moment?

Momentum of order 0 (volume)

\[ M_0(\omega) = \int_{\omega} dx = |\omega| \]

Volume fraction (relative to a cell \( \Omega \))

\[ \mu(\omega) = \frac{M_0(\omega)}{M_0(\Omega)} \]

Momentum of order 1

\[ M_1(\omega) = \int_{\omega} xdx \]

Centroid

\[ x_c(\omega) = \frac{M_1(\omega)}{M_0(\omega)} \]
Moment-of-fluid: Moment?

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\[ \boldsymbol{x}_c(\omega) = \frac{M_1(\omega)}{M_0(\omega)} \]
Moment-of-fluid: Formulation

Data:
- Volume fraction of any fluids $\mu$ in each cells
- Centroid of any fluids $x_c$ in each cells

Reconstruction method:
- VOF: $|\omega^\ell| = |\omega^*|$ for each cell
  $\rightarrow$ under-determined problem!
- MOF: $|\omega^\ell| = |\omega^*|$ and $x_c(\omega^\ell) = x_c(\omega^*)$ for each cell
  $\rightarrow$ over-determined problem!

Minimization problem:
- Find $\omega^\ell = \arg\min_{\omega^\ell} |x_c(\omega^\ell) - x_c(\omega^*)|^2$
- Under constraint $|\omega^\ell| = |\omega^*|$

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Minimization: Example

- $\mu(\omega^*) = 0.3$
- $x_c(\omega^*) = (-0.3, -0.3)$

Objective function:

$$|x_c(\omega^*)(\phi) - x_c(\omega^*)|^2$$

Solution: $\phi \approx 0.841$
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Objective function:

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Solution: $\phi \approx 0.841$
Example of static reconstruction: Zigzags

Example of static reconstruction: Compass

Example of static reconstruction: Pie

Example of static reconstruction: Stars

Analytic reconstruction: Motivations

- On Cartesian grids, cells are rectangles
- Rectangle are very simple shapes
- Upgrade a VOF algorithm
- Easier to implement
- No problem with local minima
- Faster?
Analytic reconstruction: possible configurations ($\mu \leq 0.5$)
Analytic solution: Parabola

For all \( x \in \left[ \frac{c_x}{3}, \frac{2c_x}{3} \right] \)

\[
P(x) = \frac{V}{2c_x} + \frac{6V}{c_x} \left( \frac{1}{2} - \frac{x}{c_x} \right)^2
\]

Let \( p = (p_x, p_y) \) any point of \( \mathbb{R}^2 \)

The closest point of \( p \) to the parabola \( P \) is its orthogonal projection

The coordinate \( x \) of \( x_c(\omega^\ell) = (x, P(x)) \) is one of the solution of

\[
x - p_x - \frac{12V}{c_x^2} \left( \frac{1}{2} - \frac{x}{c_x} \right) \left( \frac{V}{2c_x} - p_y \right) - \frac{72V^2}{c_x^3} \left( \frac{1}{2} - \frac{x}{c_x} \right)^3 = 0
\]
Analytic solution: Hyperbola

For all $x \in \left[ \frac{1}{3} \sqrt{2V \frac{c_x}{c_y} \frac{c_x}{3}} \right]$

$$H(x) = \frac{9V}{2x}$$

Let $p = (p_x, p_y)$ any point of $\mathbb{R}^2$

The closest point of $p$ to the hyperbola $H$ is its orthogonal projection

The coordinate $x$ of $x_c(\omega^\ell) = (x, H(x))$ is one of the solution of

$$x^4 + p_x x^3 + \frac{2}{9} V p_y x - \left( \frac{2V}{9} \right)^2 = 0$$
Analytic solution: Algorithm

1. If $\mu > 0.5$ solve the dual problem
2. Locate the quadrant where $x_c(\omega^*)$ is
   - $x_c(\omega^*) \in Q_1$ try \{1, 2, 4\}
   - $x_c(\omega^*) \in Q_2$ try \{2, 3, 6\}
   - $x_c(\omega^*) \in Q_3$ try \{6, 8, 9\}
   - $x_c(\omega^*) \in Q_4$ try \{4, 7, 8\}
3. Solve 2 cubic and 1 quartic
4. Find the closest solution
5. Compute $n$ and $d$ from the solution
Moment of Fluid: Summary

- Stencil reduced to only one cell
- Analytic reconstruction is about 30% faster than minimization
- What about multimaterial?
Multiphase reconstruction: Serial dissection

The best solution minimizes the sum of the centroid defects.

Multiphase reconstruction: Examples

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Advection

Backward advection: Compute the pre-image $\Omega^{n-1}$ of $\Omega^n$ with a Runge-Kutta 2 method.

Compute the volume of red and blue fluid.

Remark: Only the vertices of the cell are advected. The volume is not exactly preserved.
Advection: Centroids

We can show that the centroids almost follow an advection equation:

\[
\frac{d}{dt} x_c(\omega) = \mathbf{v}(x_c(\omega)) + \mathcal{O}(h^2)
\]

Advection algorithm:
- Backward advection of the cell (RK2)
- Intersection of the fluid polygons
- Compute volume and centroids
- Forward advection of the centroids (RK2)

Source: Dyadechko, V., Shashkov, M. (2007)
Advection: 2 fluids on a sheared flow
Advection: 5 fluids on a sheared flow
Advection: 5 fluids on a sheared flow
Advection: 5 fluids on a periodic flow
Advection: Focus on the B-tree dissection

Without B-tree dissection

With B-tree dissection
Numerical results: Error computation

**Local errors**

- **Distance error**
  \[
  \Delta \Gamma = \max_{x^* \in \Gamma^*} \min_{x \in \Gamma} |x - x^*|
  \]

- **Area of symmetric difference**
  \[
  \Delta \omega = |\omega^e \Delta \omega^*|
  \]
  \[
  A \Delta B = (A \setminus B) \cup (B \setminus A)
  \]

**Global error**

- **Average deviation (equivalent to \(\Delta \Gamma\))**
  \[
  \Delta \Gamma_{avg} = \frac{1}{|\partial \omega^*|} \sum_{i=1}^{N} |\omega^e_i \Delta \omega^*_i|
  \]

Numerical results: Sheared flow spatial convergence

Case description:

\[
\mathbf{u}(x, y) = \begin{bmatrix}
-2\sin^2(\pi x)\sin(\pi y)\cos(\pi y) \\
2\sin^2(\pi y)\sin(\pi x)\cos(\pi x)
\end{bmatrix} \cos \left( \pi \frac{t}{T} \right)
\]

Parameters:
- Iterations: 1000
- Time step: \(10^{-4}\) s
- Mesh: \(N \times N, N \in \{16, 32, \ldots, 4096\}\)
Numerical results: Sheared flow spatial convergence

<table>
<thead>
<tr>
<th>$N$</th>
<th>error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024</td>
<td>$3.09 \cdot 10^{-7}$</td>
<td>2.11</td>
</tr>
<tr>
<td>2048</td>
<td>$7.19 \cdot 10^{-8}$</td>
<td>2.10</td>
</tr>
<tr>
<td>4096</td>
<td>$1.60 \cdot 10^{-8}$</td>
<td>2.17</td>
</tr>
</tbody>
</table>
Numerical results: Sheared flow time convergence

Parameters:
- Total time: 0.5 s
- Time step: \( \{5 \cdot 10^{-4}, \cdots, 1.25 \cdot 10^{-4}\} \) s
- Mesh: \( 1024 \times 1024 \)

Does not converge!
Error about \( 10^{-7} \)
Numerical results: Sheared flow time convergence

Parameters:
- Total time: 0.5 s
- Time step: \( \{5 \cdot 10^{-4}, \ldots, 1.25 \cdot 10^{-4}\} \) s
- Mesh: 1024 $\times$ 1024

Order 1 convergence with Euler.
Error RK2 $<$ Error Euler

Spatial error dominates
$\Rightarrow$ Case without spatial error
Numerical results: Accelerated front flow time convergence

Velocity:

\[ u_x(x, y) = 0.3\pi \sin(\pi t) \]

Order 2 with RK2
Limitations of MOF: Filaments

- The filament does not move if the time step is too small!
Limitations of MOF: Possible solution

Initial configuration

MOF representation after advection

MOF reconstruction

- Detection of filament at the advection step
- Introduction of a virtual fluid A'
- 3-fluid reconstruction

MOF:
- 3D
- Filaments
- Analytic solutions for triangles and quadrangles

Around MOF:
- Coupling with immersed boundaries
- CLS-MOF
- Order 2 with the energy equation and Navier-Stokes
Appendix
Centroid advection

Fluid domain $\omega(t)$. Eulerian velocity $u(x, t)$. $\text{div} \, u = 0$.

\[
\frac{d}{dt} \int_{\omega(t)} x \, dx = \int_{\omega(t)} \left( \frac{\partial}{\partial t} \text{Id}(x) + u(x, t) \cdot \nabla \text{Id}(x) + \text{Id}(x) \text{div} \, u(x, t) \right) \, dx
\]

\[
= \int_{\omega(t)} u(x, t) \, dx
\]

\[
= \int_{\omega(t)} \left( u(x_c, t) + [\nabla u(x_c, t)] (x - x_c) + \mathcal{O}(|x - x_c|^2) \right) \, dx
\]

\[
= |\omega(t)| u(x_c, t) + \nabla u(x_c, t) \int_{\omega(t)} (x - x_c) \, dx + \mathcal{O}(h^2) = 0
\]

Thus

\[
\frac{d}{dt} x_c = u(x_c) + \mathcal{O}(h^2)
\]