Immersed boundary methods for anisotropic Cartesian meshes

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CPU
Le monde numérique au service de la certification et de la sécurisation des systèmes

Introduction

Archaeologists are interested in prehistoric human activities in caves:

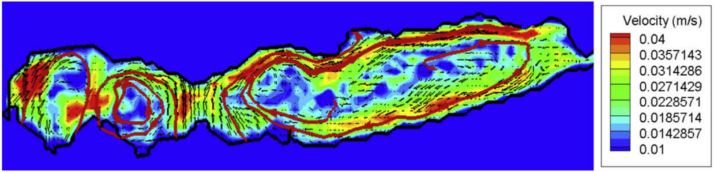
- How and why they did paintings?
- How they did fire without injuring themselves? —and why?
- etc.

Numerical simulation of cave air flows is a powerful tool:

- Cave climate
- Fire dynamics
- Preventive tool for painting preservation

Picture by: Norbet Aujoulat

Painting in Lascaux caves



D. Lacanette *et al.*, *IJHMT 52*, 2009

I2M is improving these simulations to reach archaeologists needs.

Air flow simulation in caves

Grid resolution

Today:	$\Delta x = 30\,\text{cm}$	$\rightarrow$	5×10^6 nodes	$\rightarrow$	50 procs
Objective:	$\Delta x = 5\,\text{cm}$	$\rightarrow$	1000×10^6 nodes	$\rightarrow$	10 000 procs

counting 100 000 nodes by proc

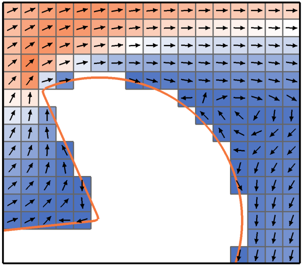
- Need for **massively parallel** solvers

Notus

New CFD code at I2M
deigned for massively parallel simulations
`http://notus-cfd.org`

Complex geometry

- Using an **immersed boundary** method
- **Can IB method be used on massively parallel solvers?**



Outline

Part 1

Adapting immersed boundary methods: Laplace equation

1.1 The Immersed Boundary method

Ghost-Fluid Finite-Differences

Domain split in two: **inner** and **outer** domains

An equation is discretized at inner nodes:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} = f_i \quad \text{(Laplace equation)}$$

some $u_{i\pm1}$ may be an outer node with no value.

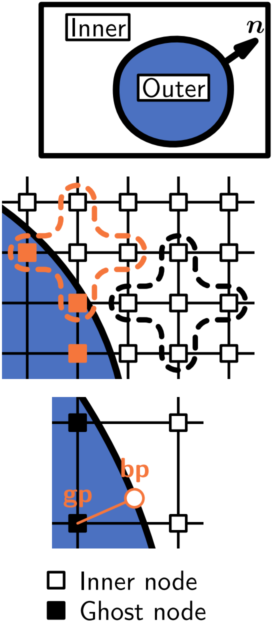
- These are the **ghost nodes**

Closure

Dirichlet or Neumann boundary conditions are applied at **boundary points (bp)**

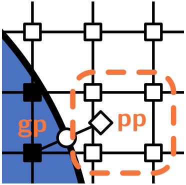
With IB, **bp** is not at a grid node:

- The boundary condition is **interpolated**



Interpolation methods

The linear method[1]

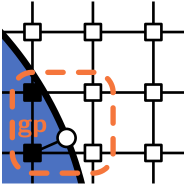


$$u_{\mathbf{bp}} = \frac{1}{2}(u_{\mathbf{pp}} + u_{\mathbf{gp}}) \qquad \text{(Dirichlet)}$$

$$\partial_{\boldsymbol{n}} u_{\mathbf{bp}} = \frac{1}{2L}(u_{\mathbf{pp}} - u_{\mathbf{gp}}) \qquad \text{(Neumann)}$$

and the **probe point** $u_{\mathbf{pp}} = \sum_j \alpha_j u_j$

The direct method[2]



$$u_{\mathbf{bp}} = \sum_j \beta_j u_j \qquad \text{(Dirichlet)}$$

$$\partial_{\boldsymbol{n}} u_{\mathbf{bp}} = \sum_j (\boldsymbol{g}_j | \boldsymbol{n}) u_j \qquad \text{(Neumann)}$$

Both methods uses level set info. $(L, \boldsymbol{n})$ at **gp**

With IB, the boundary conditions are coupled

- **The linear system is modified**

$$\underbrace{Au = f}_{\text{(Laplace equation)}} \qquad \underbrace{Eu = b}_{\text{(IB equation)}} \qquad \Rightarrow \qquad \underbrace{(A+E)u = f + b}_{\text{(Modified equation)}}$$

[1] Mittal, *et al.* "A versatile sharp interface immersed boundary method for..." *JCP 227*, n° 10, 2008

[2] Coco, A. & Giovanni R. "Finite-difference ghost-point multigrid methods on Cartesian..." *JCP 241*, 2013

1.2 Constraints of massively parallel simulations

Constraints

Using geometric multigrid solvers
e. g. Hypre SMG

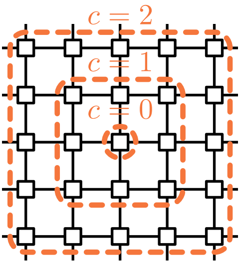
- Efficient solvers requires **compact** matrices

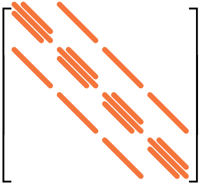
The matrix A is **compact of order** c if:

$$|\boldsymbol{x}_j - \boldsymbol{x}_i| > \Rightarrow A_{ij} = 0$$

The matrix A is a band matrix

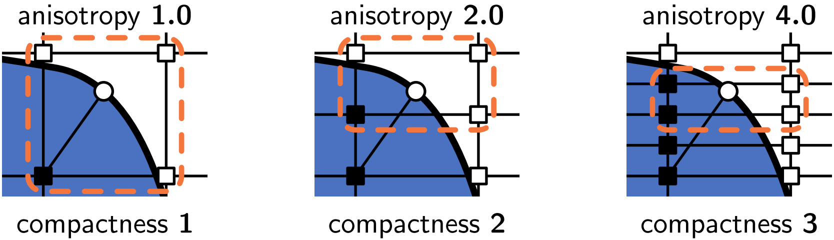
- **The IB method must produce compact matrices**





Band matrix

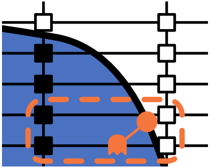
Compactness of operators



- **The unmodified IB method does not produce compact matrices**

Hence, we introduce the following modification

Ghost point shifting



The **gp** can be shifted along a grid edge such as

- the **linear** method gets **compact of order 2**
- the **direct** method gets **compact of order 1**

- **The modified IB method produces compact matrices**

1.3 An example with the Laplace equation

Solving the equation:

$$\boxed{\Delta u(x, y) = 2}$$

$$u(-1, y) = 0 \quad u(+1, 0) = 4$$

IB: circle of radius 0.65

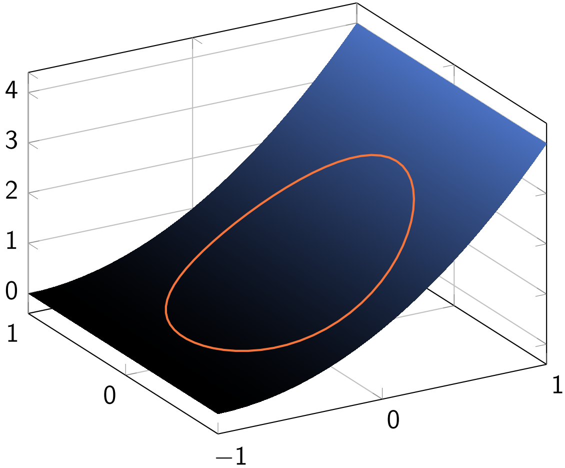
Boundary conditions:

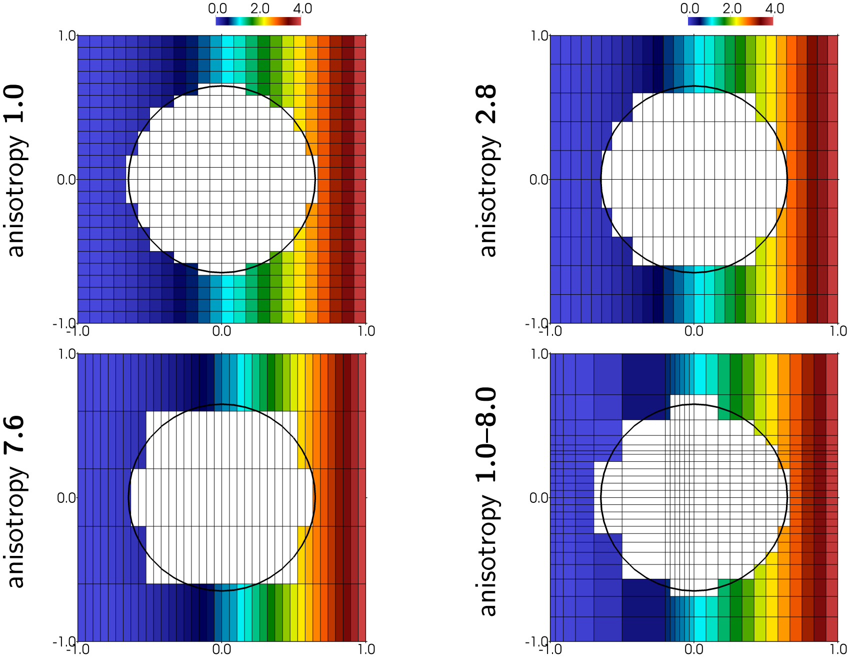
- Dirichlet
  $$u_\Gamma = (1 + x_\Gamma)^2$$
- Neumann
  $$\partial_{\boldsymbol{n}} u_\Gamma = 2(1 + x_\Gamma)$$

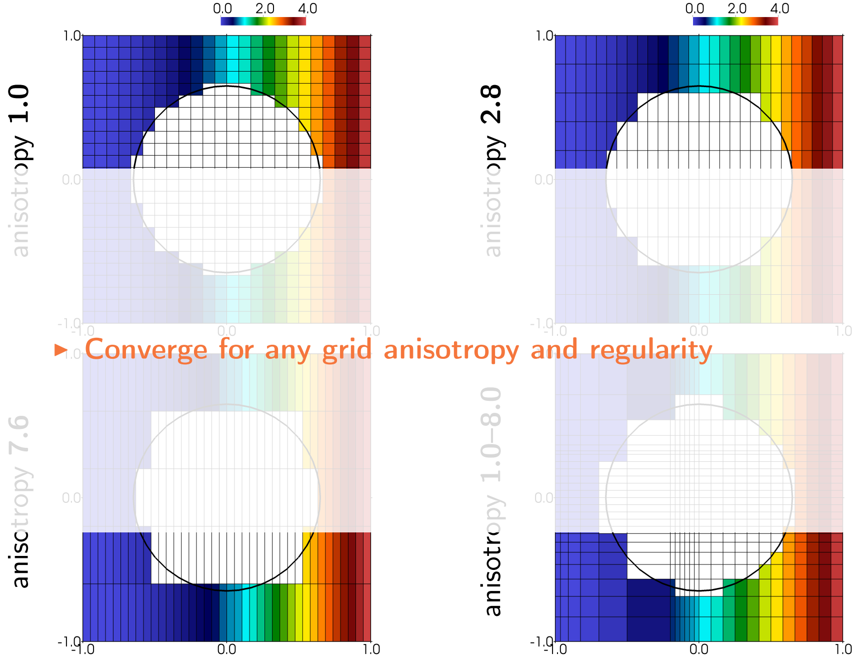
Solution:

$$u(x, y) = (1 + x)^2$$

Reference solution

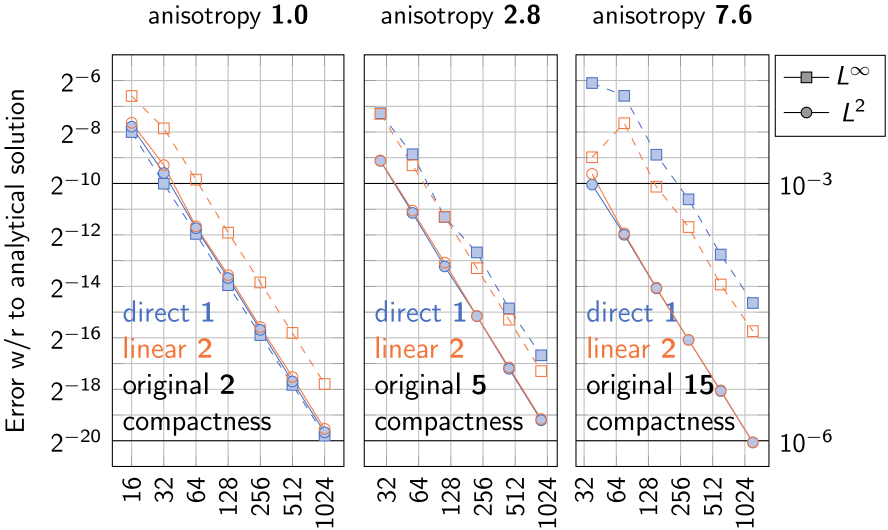



0.0 2.0 4.0
anisotropy 1.0
anisotropy 2.8
anisotropy 7.6
anisotropy 1.0–8.0
1.0
0.0
-1.0
-1.0
0.0
1.0

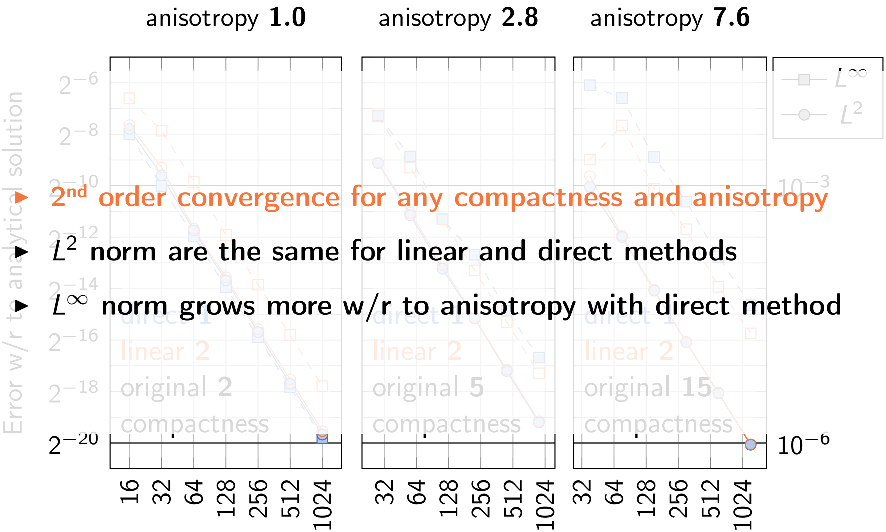


- Converge for any grid anisotropy and regularity

With **Dirichlet** boundary condition

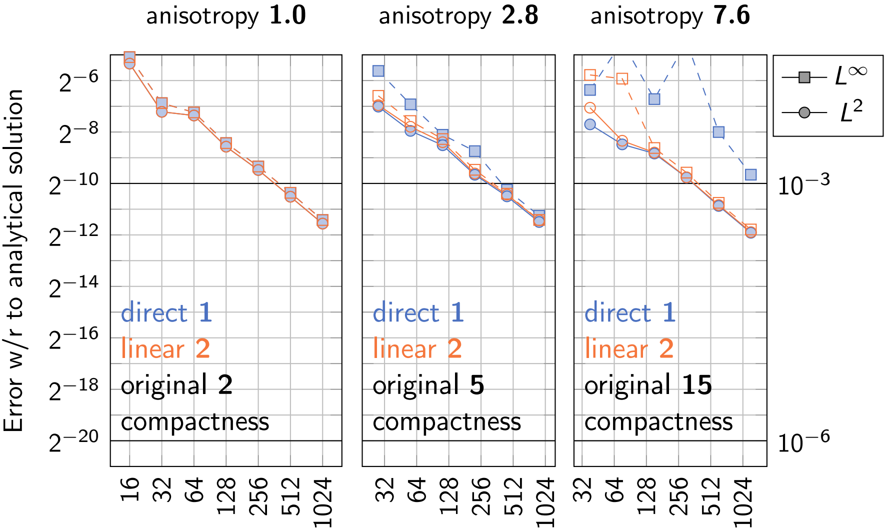


With **Dirichlet** boundary condition

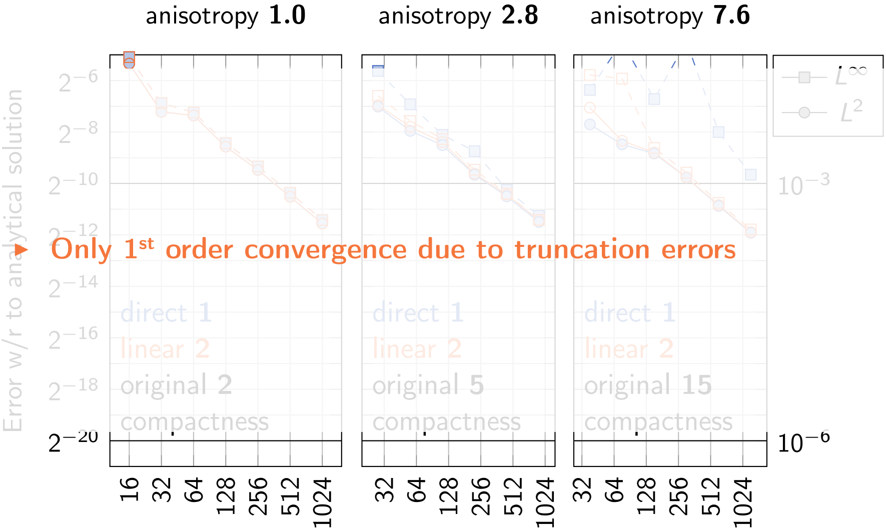


- **2nd order convergence for any compactness and anisotropy**
- **L^2 norm are the same for linear and direct methods**
- **L^∞ norm grows more w/r to anisotropy with direct method**

With **Neumann** boundary condition

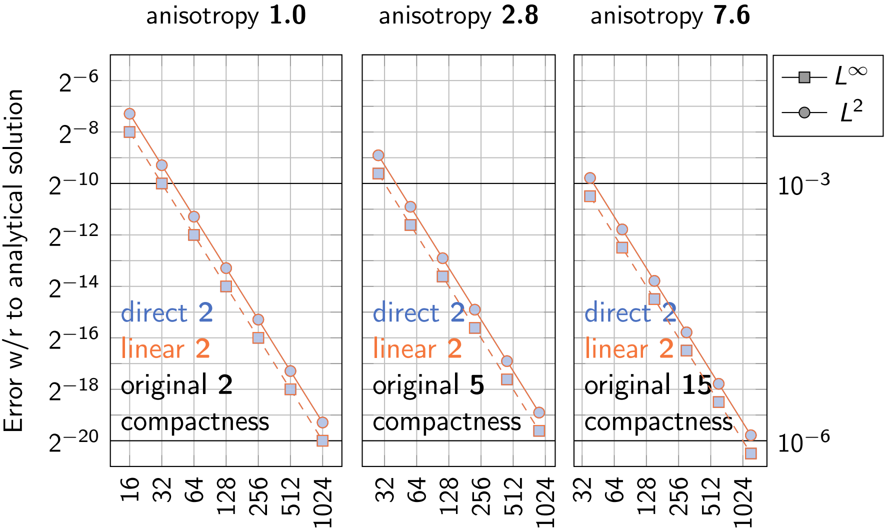


With **Neumann** boundary condition

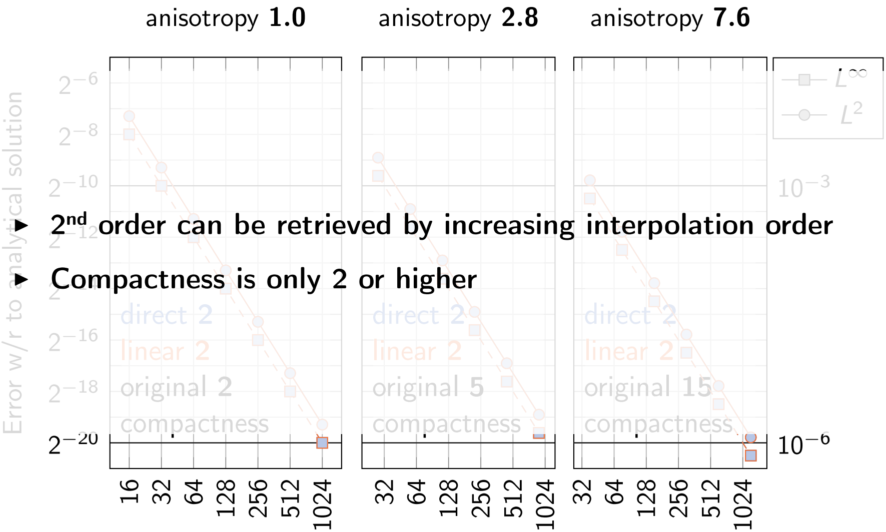


▶ Only 1st order convergence due to truncation errors

With **Neumann** boundary condition



With **Neumann** boundary condition



- **2^nd^ order can be retrieved by increasing interpolation order**
- **Compactness is only 2 or higher**

Part 2

Adapting immersed boundary methods: Navier-Stokes equations

2.1 Modification of Navier-Stokes equations

Fractional-step method

$$\boxed{\partial_t \boldsymbol{u}^* + \mathrm{div}(\boldsymbol{u}^* \otimes \boldsymbol{u}^n) + \boldsymbol{\nabla} p^n = \nu \boldsymbol{\Delta} \boldsymbol{u}^*}$$

$$\boxed{\delta t\, \Delta\Phi = \mathrm{div}\, \boldsymbol{u}^*}$$

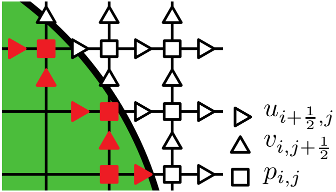
$$p^{n+1} = p^n + \Phi \qquad \boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \delta t\, \boldsymbol{\nabla}\Phi$$

- **Two linear systems for $\boldsymbol{u}^*$ and Φ**

Using a **staggered grid**:
$\boldsymbol{u} \to [u_{i+\frac{1}{2},j}, v_{i,j+\frac{1}{2}}]^\mathsf{T}$ and $p \to p_{i,j}$

Boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{(wall)}$$



The nonlinear term needs u^n **values on ghost cells**

- Using IB extrapolation as an additional velocity correction

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \delta t\, \boldsymbol{\nabla}\Phi \quad \Rightarrow \quad \begin{aligned} \boldsymbol{u}^{**} &= \boldsymbol{u}^* - \delta t\, \boldsymbol{\nabla}\Phi \\ (\boldsymbol{I} + \boldsymbol{E})\boldsymbol{u}^{n+1} &= \boldsymbol{u}^{**} + b \end{aligned}$$

2.2 Couette flow

IB: coaxial cylinders of radius

$R_1 = 0.02$ and $R_2 = 0.1$

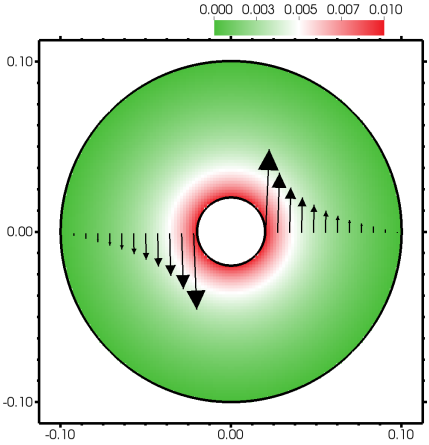
Boundary conditions:

$u(R_1) = 0.01$ and $u(R_2) = 0$

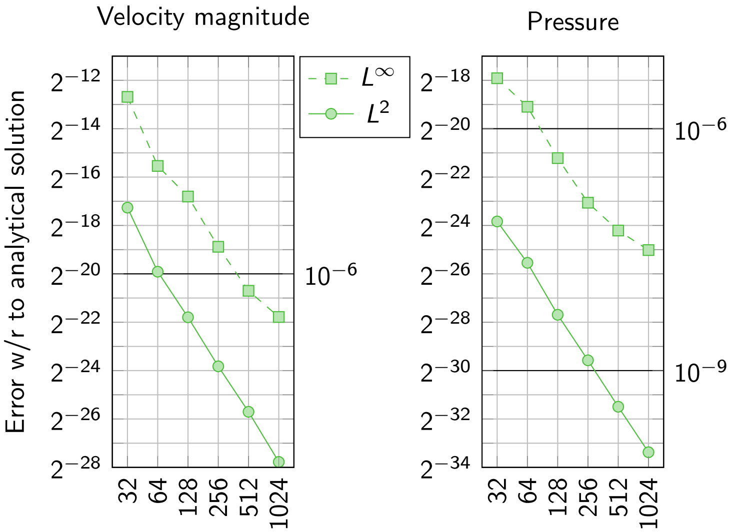
Solution:

$$\boldsymbol{u}(r,\theta) = u(r)\boldsymbol{e}_\theta$$

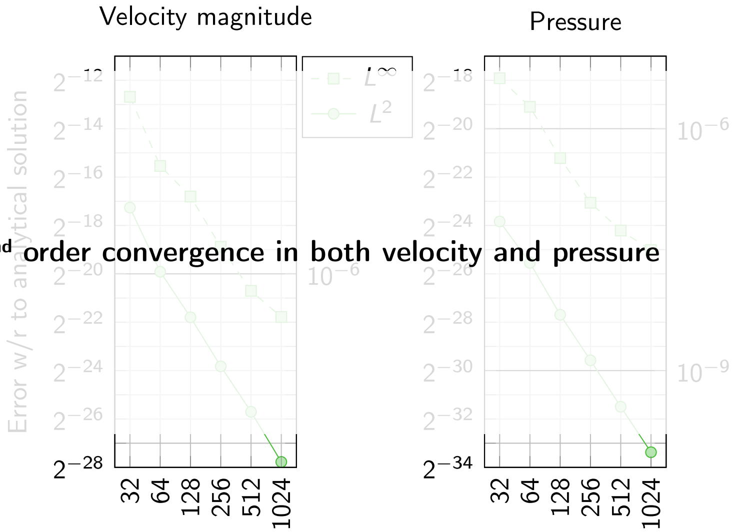
$$u(r) = \frac{A}{2}r + \frac{B}{r}$$



Stationary flow



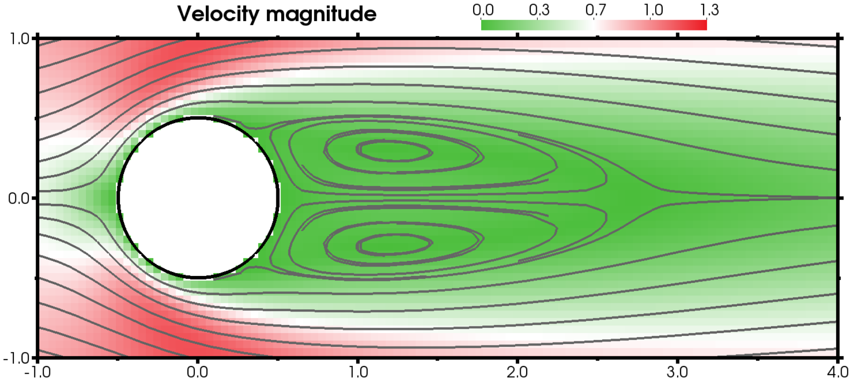
Stationary flow



- **2nd order convergence in both velocity and pressure**

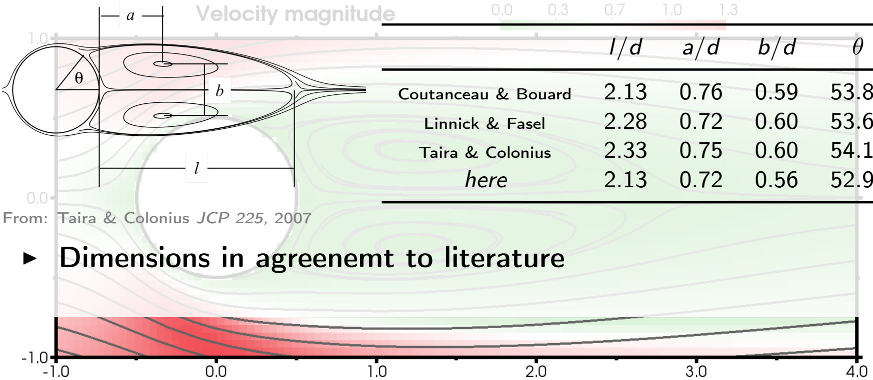
2.3 Flow around a cylinder

IB: cylinder $d = 1$
Regime: stationary $Re = 40$



2.3 Flow around a cylinder

IB: cylinder $d = 1$
Regime: stationary $Re = 40$

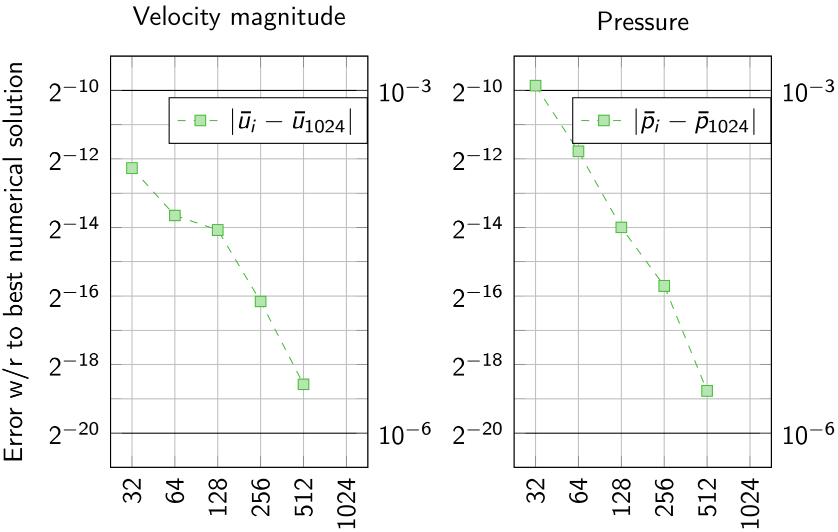


From: Taira & Colonius *JCP 225*, 2007

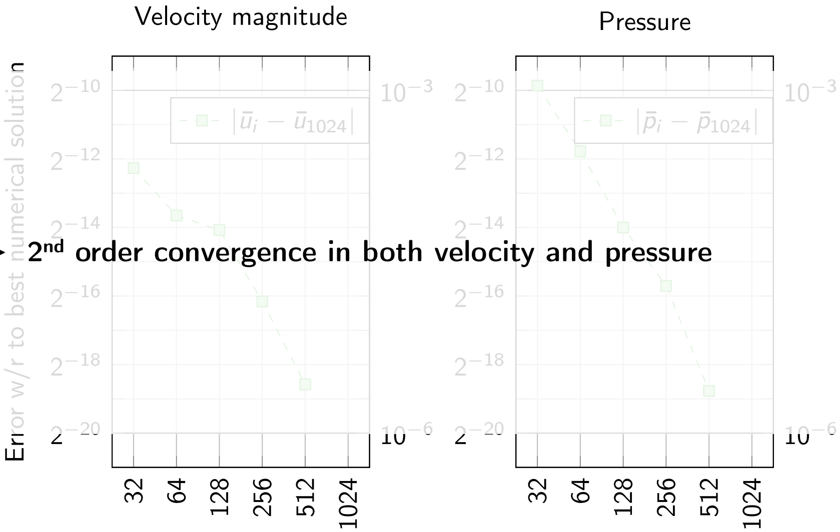
	l/d	a/d	b/d	θ
Coutanceau & Bouard	2.13	0.76	0.59	53.8°
Linnick & Fasel	2.28	0.72	0.60	53.6°
Taira & Colonius	2.33	0.75	0.60	54.1°
here	2.13	0.72	0.56	52.9°

- **Dimensions in agreenemt to literature**

Stationary flow



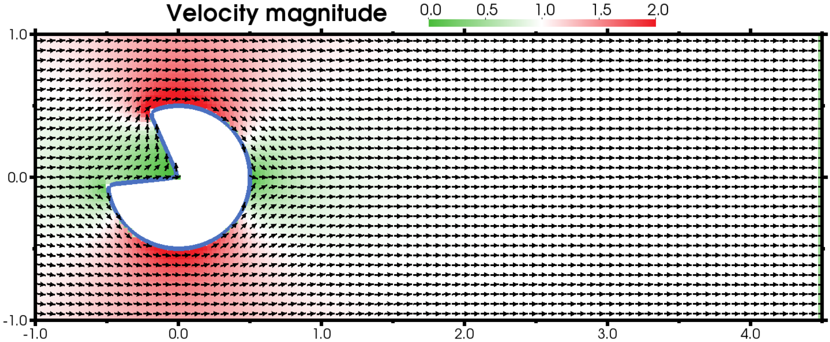
Stationary flow



- 2^{nd} order convergence in both velocity and pressure

2.4 Flow around an irregular cylinder

IB: cylinder of radius $R = 0.5$
Regime: non-stationary $Re = 60$



Conclusions

What has been done

Our **immersed boundary method**

- works on **Cartesian anisotropic meshes**
- converge at 2^{nd} order
- **is compact of order 1 (Dirichlet) or 2 (Neumann)**

Implemented and tested in Notus

What remains to do

- Benchmark on massively parallel applications
- Application to Lascaux caves